

## Symmetry broken motion of a periodically driven Brownian particle: Nonadiabatic regime

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We report a theoretical study of an overdamped Brownian particle dynamics in the presence of both a spatially modulated one-dimensional periodic potential  $U(x)$  and a periodic alternating force (AF). As the periodic potential  $U(x)$  has a low symmetry (a *ratchet potential*) the Brownian particle displays a broken symmetry motion with a nonzero time average velocity. By making use of the Green function method and a mapping to the theory of Brillouin bands the probability distribution  $P(x,t)$  of the particle coordinate  $x$  is derived and the nonlinear dependence of the macroscopic velocity on the frequency  $\omega$  and the amplitude  $\eta$  of the AF is found. In particular, our theory allows to go beyond the adiabatic limit ( $\omega=0$ ) and to explain the peculiar reversal of the velocity sign found previously in the numerical analysis.

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The model of a particle moving in a periodic one-dimensional potential  $U(x)$  has been used to explain diverse fascinating phenomena in various fields of physics, chemistry, and biology [1–3]. Well known examples of such phenomena are Brillouin bands and Bloch oscillations in the solid state theory [4], dynamics of Josephson junctions [5], energy transport in various biological systems [6], etc.

The peculiar property of a particle motion in the periodic potential is the presence of two different states, a static state and a whirling (dynamic) state. In the dynamic state, the particle has a nonzero value of the time averaged velocity  $v = \langle \dot{x}(t) \rangle$ . As the system is subject to an external noise, the “*Brownian*” particle motion becomes more complex (it displays both random damped oscillations and random jumps between the potential wells [7]), the difference between libration and rotation states disappears.

The theory of Brownian particle motion in a specific periodic potential of cos-like shape and in the presence of a constant driving force (DF) has been developed in Refs. [2,8,9]. As a particular result it was found that in the absence of a DF the mean particle velocity is zero. Moreover, it is well known [6,10] that equilibrium noise only cannot lead to the fluctuation induced transport, and correspondingly,  $v = 0$  for the particle motion in an arbitrary periodic potential  $U(x)$ . In this case the diffusive motion of particle occurs. The probabilities of fluctuation induced particle escape from the potential well to the left and to the right are identical. These probabilities are determined by the amplitude of potential  $U(x)$  and do not depend on the symmetry of potential.

The situation changes drastically when a Brownian particle is subject to both a periodic spatially modulated potential (PP) and a periodic alternating force (AF). It was shown in Ref. [11] by numerical analysis of the corresponding Fokker-Planck equation that in this case the reflection symmetry of an overdamped particle motion ( $x$  to  $-x$ ) can be broken, and directed transport occurs. The specific condition of such transport is the low symmetry of the periodic potential  $U(x)$  (a *ratchet potential*). Moreover, by making use of the symmetry arguments and the numerical analysis it was shown that directed transport occurs even in the presence of symmetric PP as the periodic AF has a low symmetry in time [12].

An analytical description of a Brownian particle motion in an anisotropic periodic potential has been carried out in an adiabatic regime as the frequency of AF  $\omega$  is rather small [11,13]. In this limit it was found for arbitrary values of the noise strength and the amplitude of AF  $\eta$ , the particle moves in the direction of the slower rate of potential change [13]. However, in Ref. [11] the peculiar *reversal of the sign* of  $v$  has been found as the AF frequency  $\omega$  becomes relatively large. Moreover, in this nonadiabatic limit and as the amplitude of AF  $\eta$  increases, the dependence  $v(\eta)$  displays oscillations. These effects do not appear in the overdamped deterministic regime where the noise strength is zero [14]. Notice here, that the directed transport of Brownian particle is more complex than the one in the deterministic (overdamped) case. In latter case, the nonzero value of  $v$  appears as a simple consequence of two different values of a depinning force, and the directed transport is absent as the amplitude of AF is small. Thus, in order to explain various peculiarities of the overdamped Brownian particle motion we develop a theoretical analysis that goes beyond an adiabatic approximation.

In this paper we present a consistent analytical approach to the dynamics of an overdamped Brownian particle in the presence of both PP  $U(x)$  with the period  $a$  and harmonic AF,  $\eta \cos(\omega t)$  [15,16]. By making use of a particular diagrammatic technique that is valid in the limit of a large time, the dependence of the mean velocity on the amplitude  $\eta$  and frequency  $\omega$  of AF will be calculated. We find that in the presence of AF the average velocity is determined by various *relaxation processes inside the potential well*, and the directed transport of Brownian particle occurs as the potential  $U(x)$  has a low symmetry. We also obtain the particular range of  $\omega$  where the interplay between noise and AF leads to the reversal of the sign of  $v$ . Although in the regime of a small  $\eta$  the direct perturbation technique also allows us to obtain the current reversal [17], the dependence of the mean velocity on the relaxation processes and, therefore, on the various parameters of the problem (i.e., strength of the fluctuations and the potential height) is not transparent in this approach.

The dynamics of an overdamped particle is described by the Langevin equation [7],

$$\alpha \dot{x}(t) + U'(x) = \eta \cos(\omega t) + \xi(t), \quad (1)$$

where the white noise function  $\xi(t)$  has a zero mean and the correlation function  $\langle \xi(t)\xi(t') \rangle = 2\alpha T \delta(t-t')$ . Here,  $\alpha$  is the damping coefficient and  $T$  is the effective temperature describing the strength of the fluctuations. Next, we introduce the time-dependent probability density  $P(x,t)$  that satisfies the Fokker-Plank equation [2,3],

$$\alpha \frac{\partial P(x,t)}{\partial t} = T \frac{\partial^2 P(x,t)}{\partial x^2} + \frac{\partial}{\partial x} \{ [U'(x) - \eta \cos(\omega t)] P(x,t) \} \quad (2)$$

with the initial condition

$$P(x,0) = \delta(x-x_0). \quad (3)$$

Using the Laplace transformation  $P_\lambda(x) = \int_0^\infty dt P(x,t) e^{-\lambda t}$  and the standard substitution  $P_\lambda(x, x_0) = \exp\{-[U(x) - U(x_0)]/2T\} G_\lambda(x, x_0)$  we obtain the integral equation for the Green function  $G_\lambda(x, x_0)$

$$G_\lambda(x, x_0) = G_\lambda^0(x, x_0) + \int \frac{d\lambda_1}{2\pi i} \eta \frac{2\lambda_1}{\omega^2 + \lambda_1^2} \frac{1}{\alpha} \int dy G_\lambda^0(x, y) e^{U(y)/2T} \frac{d}{dy} [e^{-U(y)/2T} G_{\lambda-\lambda_1}(y, x_0)], \quad (4)$$

where  $G_\lambda^0(x, x_0)$  is the Green function of the equation,

$$T \frac{d^2}{dx^2} G_\lambda^0(x, x_0) + G_\lambda^0(x, x_0) \times \left( -T e^{U(x)/2T} \frac{d^2}{dx^2} e^{-U(x)/2T} - \alpha \lambda \right) = -\alpha \delta(x-x_0). \quad (5)$$

The mean value of particle velocity  $v$  is determined by the probability  $P(x,t)$  in the limit of large time,

$$v = \lim_{t \rightarrow \infty} \int dx P(x,t) \frac{(x-x_0)}{t}. \quad (6)$$

In the absence of AF ( $\eta=0$ ) it immediately follows from Eq. (5) and the property [18]  $G_\lambda^0(x, x_0) = G_\lambda^0(x_0, x)$  that the mean value of the velocity is zero for an arbitrary periodic potential  $U(x)$ . However, in the presence of AF the probability density  $P(x,t)$  is determined by the product of the Green functions with different arguments  $\lambda$ . It leads to the breaking of the symmetry and, therefore, to the directed transport.

The perturbation series over the amplitude of AF  $\eta$  can be presented in the diagrammatic form (see, Fig. 1). Moreover, we are interested in a nonoscillating part of  $P(x,t)$  only. Thus, the directed transport is determined by *even* order terms only. The zero, second, and fourth order terms are shown in Fig. 1. The arguments  $\lambda$  and  $p$  of the Green function  $G_\lambda^0(p, x, x_0)$  and  $P_\lambda^0(p, x, x_0)$  written in the mixed momentum-coordinate representation are determined corre-

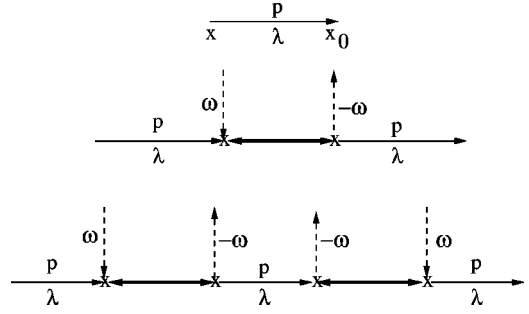


FIG. 1. The most important diagrams and the diagrammatic presentation of Eq. (4). Thin and thick solid lines correspond accordingly to  $G_\lambda^0(p; x, y)$  and  $P_\lambda(x, y)$ ; dashed lines are due to the presence of AF. The cross presents an operator  $(\eta/\alpha)(d/dx)$ . In the limit of large time the parameters  $p$  and  $\lambda$  are small.

spondingly by the conservation of “energy” and “quasimomentum” in the knots. The integrals over all intermediate coordinates have to be taken.

To obtain the fluctuation induced transport in the presence of AF we need to know the long time behavior of the probability distribution  $P(x,t)$  [see, the Eq. (6)]. This limit corresponds to small values of  $\lambda$  in the integral Eq. (4). Physically it may be interpreted as the particle return to the stable state after the multiple interactions with AF. It allows greatly to simplify the Eq. (4) just keeping most singular contributions with small  $\lambda$  arguments of the Green functions  $G_\lambda^0(x, y)$ . These terms can be presented in the diagrammatic form (see, Fig. 1). By summing these terms and calculating the intermediate integrals over  $\lambda_n$  we obtain the expression for  $G_\lambda(x, x_0)$  in the form

$$G_\lambda(x, x_0) = \int dp e^{ip(x-x_0)} G_\lambda^0(p; x, x_0) \frac{Z_\lambda(p)}{1-Z_\lambda(p)}. \quad (7)$$

Here, we introduce the *periodic* Green function  $G_\lambda^0(p; x, x_0)$  written in the mixed momentum-coordinate representation, and  $Z_\lambda(p)$  is determined by expression,

$$Z_\lambda(p) = \frac{\eta^2}{\alpha^2} \int dy_1 dy_2 P_\lambda^0(p; y_2, y_1) \frac{d}{dy_1} \frac{d}{dy_2} \text{Re } P_{i\omega}(y_1, y_2). \quad (8)$$

To calculate the function  $Z_\lambda(p)$  and correspondingly, to obtain the conditions when the directed transport occurs, we use the general properties of the Eq. (5). This equation can be mapped to a well known problem of an electron motion in a periodic potential [3,4]. Thus, the relaxation times spectrum  $\alpha/\tau_n(p)$  of the Eq. (5) contains an infinite number of bands and is determined by the wave vector  $p$ . The lowest band starts from the zero value, and in the region of small wave vectors  $p$  has a form

$$\frac{1}{\tau_n(p)} = Dp^2, \quad p \ll \frac{2\pi}{a}, \quad (9)$$

where the *diffusion coefficient*  $D$  depends on the strength of the potential and temperature. In the limit of small fluctuations as the amplitude of  $U(x)$  is large and the temperature  $T$  is small, we obtain that the value of  $D \propto \exp(-U_{max} - U_{min}/T) = e^{-\Delta U/T}$  is also small. Note here, that the diffusion coefficient  $D$  does not depend on the symmetry of the potential  $U$  and is determined by the narrow regions close to the top and bottom of potential. In the absence of AF the particle displays a diffusive motion and it is completely determined by  $D$ . The lowest band can also be mapped to a well known tight-binding model [4]. The corresponding eigenfunction  $\psi_p(x)$  has the form of a Bloch wave [4], namely,  $\psi_p(x) = e^{ipx} r_p(x)$ , where the periodic function

$$r_p(x) \approx \exp\left[-\frac{U(x)}{2T}\right] + p u_0(x) \quad (10)$$

in the limit of small  $p$ . Here, the first correction  $u_0(x)$  in the region of small  $p$  has been found in Refs. [3,19]. Upper bands correspond to much smaller relaxation times  $\tau_n$  but exactly these bands determine the directed transport.

In the limit of large time the main contribution to the probability distribution  $P(x,t)$  results from a small momentum  $p$ . The presence of an *odd* component of  $Z_\lambda(p) \propto p$  leads to a nonzero value of  $v$ , and thus, by using Eqs. (9) and (10) and substituting Eqs. (8) and (7) in Eq. (6) we arrive at the expression for  $v$

$$v = \frac{\eta^2}{\alpha^2} \frac{2a}{3A} \int_0^a dy_1 \int_0^a dy_2 \exp\left(\frac{-3U(y_1)}{2T}\right) \exp\left(\frac{3U(y_2)}{2T}\right) \frac{d}{dy_1} \text{Re } \tilde{G}_{i\omega}(y_1, y_2), \quad (11)$$

where  $A = \int_0^a dy_1 \int_0^a dy_2 e^{-U(y_1)/T} e^{U(y_2)/T}$ . Here, the Green function  $\tilde{G}_\lambda(x, x_0)$  is a solution of the Eq. (5) with periodic boundary conditions.

The formula (11) is convenient for analysis of both general properties of Brownian particle motion and particular limits. Thus, a general feature of an overdamped Brownian particle transport is that the average velocity  $v$  is determined by the properties of potential  $U(x)$  on the *whole range* of its variation. Moreover, the directed transport of Brownian particle is an interplay of two effects: the fluctuation induced particle escape from the potential well [the coefficient  $1/A$  in the Eq. (11)] and the AF induced relaxation processes inside the potential well [the Green function  $\tilde{G}_{i\omega}(y_1, y_2)$  in the Eq. (11)]. However, if the potential  $U(x)$  displays a reflection symmetry, namely,  $U(x)$  is an even function of  $x$ , the average value of  $v$  vanishes due to a fact that  $(d/dy_1) P_{i\omega}(y_1, y_2)$  is an odd function of  $y_1$  or  $y_2$ , and one of the integrals in the Eq. (11) is zero.

If the potential  $U(x)$  has no reflection symmetry, the average velocity is not zero, and it increases as  $\eta^2$  in the limit of a small amplitude of AF  $\eta$ . In this limit by making use of the eigen functions  $\psi_n(x)$  of the Eq. (5) we get

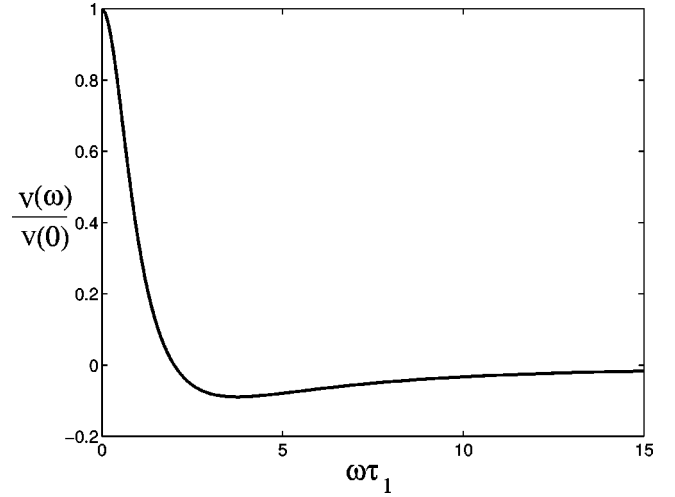


FIG. 2. The frequency dependence of the mean value of velocity  $v$ : the limit of large effective temperature.

$$\langle v \rangle = \frac{\eta^2}{\alpha^2} \frac{2a}{3A} \sum_n \frac{\tau_n}{1 + (\omega \tau_n)^2}$$

$$\int_0^a dy_1 \int_0^a dy_2 \exp\left(\frac{-3U(y_1)}{2T}\right) \exp\left(\frac{3U(y_2)}{2T}\right) \psi_n(y_2) \psi_n'(y_1). \quad (12)$$

Thus, the average velocity  $v$  is small in both limits of small and large effective temperature  $T$

$$v(T) \propto \begin{cases} e^{-\frac{\Delta U}{T}}, & T \ll U_{max} - U_{min} \\ T^{-3} f(\omega \tau_1), & T \gg U_{max} - U_{min}. \end{cases} \quad (13)$$

The particular function  $f$  depends on the ratio between the frequency  $\omega$  and  $\tau_1^{-1}$  where  $\tau_1$  is the maximum relaxation time of the particle in the potential  $U(x)$ . The Eq. (12) shows that the adiabatic regime is valid only in the limit of  $\omega \ll 1/\tau_1$ . The relaxation time  $\tau_1$  depends crucially on the effective temperature  $T$ . Thus, as the fluctuations are weak ( $T \ll \Delta U$ ),  $\tau_1 = \alpha/\omega_0^2$ , where  $\omega_0$  is the frequency of small oscillations at the bottom of potential well [20]. Moreover,  $\tau_1$  can be even larger in the limit of small fluctuations as the potential  $U(x)$  has a form of a double potential well [20,21]. In the opposite regime of large fluctuations ( $T \gg \Delta U$ ) the relaxation time depends on  $T$  and decreases as

$$\tau_1 = \frac{\alpha a^2}{4\pi^2 T}. \quad (14)$$

In the limit of a small frequency  $\omega$  the main contribution to the directed transport results from the first “relaxation time” band. It leads to a particular sign of  $v$  that does not change with the temperature. However, as the frequency increases ( $\omega \gg \tau_1^{-1}$ ) the upper relaxation time bands dominate and the sign reversal can occur. As  $T$  increases the relaxation time  $\tau_1$  decreases and the adiabatic limit is recovered.

This general scenario can be verified in the limit of large temperature  $T \gg \Delta U$  for a particular model of low symmetry potential  $U(x) = U_0(\cos(2\pi x/a) + \gamma \sin(4\pi x/a))$ , where the parameter  $\gamma$  is of order one. As the temperature is large the eigen functions  $\psi_n(x)$  of the Eq. (5) are plain waves, the main contribution to the  $v$  results from the first and second relaxation time bands, and we obtain

$$v(\omega) = \gamma \frac{\eta^2}{\alpha^2 a} \frac{U_0^3}{T^3} \tau_1 \left[ \frac{1}{1 + (\omega \tau_1)^2} - \frac{4}{16 + (\omega \tau_1)^2} \right], \quad (15)$$

where  $\tau_1$  is determined by the Eq. (14). The dependence of  $v(\omega)$  that displays a sign reversal, is presented in Fig. 2.

In the discussion presented to this point we assume that the amplitude of AF  $\eta$  is small. As  $\eta$  increases the peculiar oscillations appear in the dependence of  $v(\eta)$ . The period of these oscillations  $\Delta \eta$  can be estimated in the same limit of large temperature  $T$ . The Green function  $\tilde{G}_{i\omega}(y_1, y_2)$  can be found for arbitrary values of amplitude of AF  $\eta$  in the limit of a small potential  $U$  by making use of the Eq. (2). Substituting obtained expression in the Eq. (11) we find

$$\Delta \eta(T) \propto \begin{cases} \alpha a \omega, & \omega \gg \tau_1^{-1} \\ \alpha a \tau_1^{-1}, & \omega \ll \tau_1^{-1}. \end{cases} \quad (16)$$

In conclusion, we have presented an analysis of an overdamped Brownian particle motion in the presence of periodic potential and AF. We have shown that the directed transport in such a system is determined by two effects: the particle escape from a potential well and the relaxation processes inside the potential well. It is at variance with both the standard diffusive motion in the absence of AF and the nonlinear dynamic response of the system to a weak AF [7,20,21]. We argue that the  $\omega$  dependent reversal of the sign of  $v$  found previously by numerical analysis is due to different strength and symmetry of relaxation processes. This effect of reversal of Brownian motion should be directly observable in recently proposed Josephson junction systems [22,23]. Moreover, the observation of the dependence of directed transport on the frequency  $\omega$  and amplitude  $\eta$  of AF, allows to investigate the various relaxation processes of an overdamped Brownian particle. The used method of taking into account the most singular diagrams where the momentum  $p$  and parameter  $\lambda$  are small, can be also useful in order to analyze the fluctuation induced transport in the presence of more complex (periodic or random) AF [12,15,24].

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